Direction Cosines & Lines

1 Mark Questions

1. Write the distance of a point $P(a, b, c)$ from X-axis. **Delhi 2014C**

> Let any point on X-axis be $Q(x, 0, 0)$. Then, use the formula for distance of point $R(x_1, y_1, z_1)$ from $S(x_2, y_2, z_2)$
> $$=\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Given point is $P(a, b, c)$.

Let the coordinates of the point on X-axis be (a,0,0). **(1/2)**

[$\because$ x-coordinate of both points will be same]

$\therefore$ Required distance

$$= \sqrt{(a - a)^2 + (0 - b)^2 + (0 - c)^2}$$

$$= \sqrt{0 + b^2 + c^2}$$

$$= \sqrt{b^2 + c^2}$$ **(1/2)**

2. If the cartesian equation of a line is $\frac{3-x}{5}=\frac{y+4}{7}=\frac{2z-6}{4}$, then write the vector equation for the line. **All India 2014**

Given cartesian equation of a line is

$$\frac{3-x}{5}=\frac{y+4}{7}=\frac{2z-6}{4}$$

On rewriting the given equation in standard form, we get

$$\frac{x-3}{-5}=\frac{y+4}{7}=\frac{z-3}{2}=\lambda \quad \text{[let]}$$

$\Rightarrow$ $x=-5\lambda+3,\ \ y=7\lambda-4$

and $z=2\lambda+3$ **(1/2)**

Now, $x\hat{i}+y\hat{j}+z\hat{k}$

$$=(-5\lambda+3)\hat{i}+(7\lambda-4)\hat{j}+(2\lambda+3)\hat{k}$$

$$=3\hat{i}-4\hat{j}+3\hat{k}+\lambda(-5\hat{i}+7\hat{j}+3\hat{k})$$

$$\Rightarrow \vec{r}=(3\hat{i}-4\hat{j}+3\hat{k})+\lambda(-5\hat{i}+7\hat{j}+3\hat{k})$$

which is the required equation of line in vector form. **(1/2)**

3. Write the equation of the straight line through the point (α, β, γ) and parallel to Z-axis.

The vector equation of a line parallel to Z-axis is $\vec{m} = 0\hat{i} + 0\hat{j} + \hat{k}$. Then, the required line passes through the point $A(\alpha, \beta, \gamma)$ whose position vector is $\vec{r_1} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{j}$ and is parallel to the vector $\vec{m} = (0\hat{i} + 0\hat{j} + \hat{k})$. **(1/2)**

$\therefore$ The equation is $\vec{r} = \vec{r_1} + \lambda\vec{m}$

$$= (\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}) + \lambda(0\hat{i} + 0\hat{j} + \hat{k})$$

$$= (\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}) + \lambda(\hat{k})$$ **(1/2)**

4. Find the direction cosines of the line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$. **Delhi 2013C**

Given, equation of line is $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$

It can be rewritten as $\frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3}$

Here, DR's of the line are $-2, 6, -3$.

Now, $\sqrt{(-2)^2 + 6^2 + (-3)^2}$

$$= \sqrt{4 + 36 + 9}$$

$$= \sqrt{49} = 7 \text{ units}$$

$\therefore$ DC's of a line are $-\frac{2}{7}, \frac{6}{7}, -\frac{3}{7}$. **(1)**

5. If a unit vector $\hat{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find the value of θ. **Delhi 2013**

Given unit vector $\vec{a}$ makes angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and θ $\hat{k}$. So, $\alpha = \frac{\pi}{3}$ with $\hat{i}$, $\beta = \frac{\pi}{4}$ with $\hat{j}$ and $\gamma = \theta$ with $\hat{k}$.

Now, $\cos^2 \frac{\pi}{3} + \cos^2 \frac{\pi}{4} + \cos^2 \theta = 1$

$$[\because \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1]$$

$\Rightarrow \quad \frac{1}{4} + \frac{1}{2} + \cos^2 \theta = 1$

$\Rightarrow \quad \cos^2 \theta = 1 - \frac{3}{4} = \frac{1}{4}$ **(1/2)**

$\Rightarrow \quad \cos \theta = \pm \frac{1}{2}$

$\Rightarrow \cos\theta = \frac{1}{2}$, as θ is an acute angle.

$\therefore \quad \theta = \frac{\pi}{3}$ **(1/2)**

6. Find the cartesian equation of the line which passes through the point $(-2, 4, -5)$ and is parallel to the line $\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$.

HOTS; Delhi 2013

If two lines are parallel, then they both have same direction ratios. Use this result and simplify it.

Given, the required line is parallel to the line

$$\frac{x+3}{3}=\frac{4-y}{5}=\frac{z+8}{6} \text{ or } \frac{x+3}{3}=\frac{y-4}{-5}=\frac{z+8}{6}$$

∴ DR's of both lines are proportional to each other. **(1/2)**

The required equation of the line passing through (−2, 4, − 5) having DR's (3, − 5, 6) is

$$\frac{x+2}{3}=\frac{y-4}{-5}=\frac{z+5}{6}.$$ **(1/2)**

7. If a line has direction ratios 2, − 1, − 2, then what are its direction cosines? **Delhi 2012**

Given, DR's of the line are 2, −1, −2.

∴ Direction cosines of the line are

$$=\frac{2}{\sqrt{(2)^2+(-1)^2+(-2)^2}}, \frac{-1}{\sqrt{(2)^2+(-1)^2+(-2)^2}}, \frac{-2}{\sqrt{(2)^2+(-1)^2+(-2)^2}}$$

$$\left[\because l=\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m=\pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n=\pm\frac{c}{\sqrt{a^2+b^2+c^2}}\right]$$

$$=\frac{2}{\sqrt{4+1+4}}, \frac{-1}{\sqrt{4+1+4}}, \frac{-2}{\sqrt{4+1+4}}$$

$$=\frac{2}{\sqrt{9}}, \frac{-1}{\sqrt{9}}, \frac{-2}{\sqrt{9}}, \text{ i.e. } \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$$ **(1)**

8. What are the direction cosines of a line which makes equal angles with the coordinate axes? **Foreign 2011; All India 2009, 2008C**

Given, line makes equal angles with coordinate axes. Let α, β and γ be the angle made by the line with coordinate axes.

Than, $\alpha = \beta = \gamma \Rightarrow \cos\alpha = \cos\beta = \cos\gamma$

$\Rightarrow \quad l = m = n$ ...(i)

$[\because l = \cos\alpha, m = \cos\beta, n = \cos\gamma]$

We know that, $l^2 + m^2 + n^2 = 1$

$\therefore \quad l^2 + l^2 + l^2 = 1$ [from Eq.(i)]

$\Rightarrow \quad 3l^2 = 1 \Rightarrow l^2 = \frac{1}{3} \Rightarrow l = \pm\frac{1}{\sqrt{3}}$

From Eq. (i), direction cosines of a line are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$. **(1)**

9. Write the vector equation of the line given by $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$. **Delhi 2011,2010**

Given equation of line in cartesian form is

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}.$$

The point on the line is (5, – 4, 6) and DR's are (3, 7, 2).

We know that, vector equation of a line , if point is $\vec{a}$ and direction of a line is $\vec{b}$, is $\vec{r} = \vec{a} + \lambda\vec{b}$.

Here, $\vec{a} = (5, -4, 6)$ and $\vec{b} = (3, 7, 2)$.

So, equation of line in vector form is

$$\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$$ **(1)**

10. Equation of line is $\frac{4-x}{2}=\frac{y+3}{2}=\frac{z+2}{1}$.

Find the direction cosines of a line parallel to above line. **HOTS; All India 2011C**

Given equation of line can be written as

$$\frac{x-4}{-2}=\frac{y+3}{2}=\frac{z+2}{1}$$

Here, DR's of a line are –2, 2, 1.

∴ DC's of line parallel to above line are

given by $\frac{-2}{\sqrt{(-2)^2+(2)^2+(1)^2}}$,

$$\frac{2}{\sqrt{(-2)^2+(2)^2+(1)^2}}, \frac{1}{\sqrt{(-2)^2+(2)^2+(1)^2}}$$

or $\frac{-2}{\sqrt{4+4+1}}, \frac{2}{\sqrt{4+4+1}}, \frac{1}{\sqrt{4+4+1}}$

or $\frac{-2}{\sqrt{9}}, \frac{2}{\sqrt{9}}, \frac{1}{\sqrt{9}}$ or $-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$.

Hence, required DC's of a line parallel to the given line are $\left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$. **(1)**

NOTE Before we can use the DR's of a line, first we ensure that coefficients of x, y and z are unity with positive sign.

11. If the equations of line *AB* is $\frac{3-x}{1}=\frac{y+2}{-2}=\frac{z-5}{4}$, then write the direction ratios of the line parallel to above line *AB*. **Delhi 2011C**

Given equation of line can be written as

$$\frac{x-3}{-1}=\frac{y+2}{-2}=\frac{z-5}{4}$$

$\therefore$ DR's of the line parallel to above line are $-1, -2, 4$.

[$\because$ parallel lines have same DR's] **(1)**

12. Find the distance of point (2, 3, 4) from X-axis.
Delhi 2010C

Do same as Que. 1. **[Ans.** 5]

13. Write the equation of line parallel to the line $\frac{x-2}{-3}=\frac{y+3}{2}=\frac{z+5}{6}$ and passing through point (1, 2, 3). **All India 2009C**

Do same as Que. 6. $\left[\textbf{Ans. } \frac{x-1}{-3}=\frac{y-2}{2}=\frac{z-3}{6}\right]$

14. Write the direction cosines of a line parallel to the line $\frac{3-x}{3}=\frac{y+2}{-2}=\frac{z+2}{6}$. **Delhi 2009C**

Do same as Que. 10. $\left[\textbf{Ans. } \frac{-3}{7}, \frac{-2}{7}, \frac{6}{7}\right]$

15. The equation of line is

$$\frac{2x-5}{4}=\frac{y+4}{3}=\frac{6-z}{6}.$$

Find the direction cosines of the line parallel to this line. **All India 2008**

Do same as Que. 10. $\left[\textbf{Ans. } \frac{2}{7}, \frac{3}{7}, \frac{-6}{7}\right]$

16. The equation of line is given by $\frac{4-x}{2}=\frac{y+3}{5}=\frac{z+2}{6}$. Write the direction cosines of the line parallel to above line.
Delhi 2008C

Do same as Que. 10. $\left[\textbf{Ans. } \frac{-2}{\sqrt{65}}, \frac{5}{\sqrt{65}}, \frac{6}{\sqrt{65}}\right]$

17. If $P = (1, 5, 4)$ and $Q = (4, 1, -2)$, then find the direction ratios of PQ. **All India 2008**

Given points are $P(1, 5, 4)$ and $Q(4, 1, -2)$.

$\therefore$ Direction ratios of $PQ = 4 - 1, 1 - 5, -2 - 4$

$= 3, -4, -6$ **(1)**

$\left[\because \text{DR's of line joining points } P(x_1, y_1, z_1) \text{ and } Q(x_2, y_2, z_2) \text{ are } x_2 - x_1, y_2 - y_1, z_2 - z_1.\right]$

4 Marks Questions

18. Show that the lines $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ and $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$ intersect. Also, find their point of intersection. **Delhi 2014**

Given lines can be rewritten as

$$\vec{r} = (3\lambda + 1)\,\hat{i} + (1 - \lambda)\,\hat{j} - \hat{k} \quad \ldots\text{(i)}$$

and $\quad \vec{r} = (4 + 2\mu)\,\hat{i} + 0\hat{j} + (3\mu - 1)\,\hat{k}$ ...(ii) **(1)**

Both lines intersect at a point, when their respective components along $\hat{i}, \hat{j}$ and $\hat{k}$ are equal.

$\therefore \quad 3\lambda + 1 = 4 + 2\mu$

$\Rightarrow \quad 3\lambda - 2\mu = 3$ ...(iii)

$1 - \lambda = 0$ ...(iv)

and $\quad 3\mu - 1 = -1$ ...(v) **(1)**

From Eq. (iv), we get $\lambda = 1$ and put the value of λ in Eq. (iii), we get

$$3(1) - 2\mu = 3$$

$$\Rightarrow \quad -2\mu = 3 - 3$$

$$\Rightarrow \quad \mu = 0$$

On putting the value of μ in Eq. (v), we get

$$3(0) - 1 = -1 \Rightarrow 0 - 1 = -1$$

$$\Rightarrow \quad -1 = -1, \text{ which is true}$$

So, both lines intersect each other. **(1)**

Point of intersection of both lines can be obtained by putting $\lambda = 1$ in Eq. (i), then we get $\vec{r} = 4\hat{i} + 0\hat{j} - \hat{k}$, which is the position vector of the point of intersection $(4, 0, -1)$. **(1)**

19. Find the direction cosines of the line $\frac{x+2}{2} = \frac{2y-7}{6} = \frac{5-z}{6}$. Also, find the vector equation of the line through the point $A(-1, 2, 3)$ and parallel to the given line.

Delhi 2014C

Given equation of line is

$$\frac{x+2}{2}=\frac{2y-7}{6}=\frac{5-z}{6}$$

This equation can be written as

$$\frac{x+2}{2}=\frac{y-7/2}{3}=\frac{z-5}{-6}$$

So, direction ratio's of line are 2, 3, – 6. **(1)**

Now, direction cosines of a line are

$$l=\frac{2}{\sqrt{2^2+3^2+(-6)^2}}, \quad m=\frac{3}{\sqrt{2^2+3^2+(-6)^2}},$$

$$n=\frac{-6}{\sqrt{2^2+3^2+(-6)^2}}$$

$$\left[\because l=\frac{\pm a}{\sqrt{a^2+b^2+c^2}}, m=\frac{\pm b}{\sqrt{a^2+b^2+c^2}}, n=\frac{\pm c}{\sqrt{a^2+b^2+c^2}}\right]$$

(1)

$$\Rightarrow l=\frac{2}{\sqrt{49}}, m=\frac{3}{\sqrt{49}}, h=\frac{-6}{\sqrt{49}}$$

So, direction cosines of given line are $\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}$.

Now, DR's of a line parallel to given line are 2, 3, –6 and it passes through the point A (–1, 2, 3). So, required equation of line parallel to given line is

$$\frac{x+1}{2}=\frac{y-2}{3}=\frac{z-3}{-6}$$ **(1)**

20. Find the angle between the lines

$$\vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda(3\hat{i}+2\hat{j}+6\hat{k})$$

and $\vec{r}=7\hat{i}-6\hat{j}-6\hat{k}+\mu(\hat{i}+2\hat{j}+2\hat{k})$.

Foreign 2014; All India 2008 C

If vector form of lines are $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \lambda \vec{b_2}$, then angle between them is

$$\cos\theta = \frac{|\vec{b_1} \cdot \vec{b_2}|}{|\vec{b_1}||\vec{b_2}|}$$

The given equations of lines are

$$\vec{r} = (2\hat{i} - 5\hat{j} + \hat{k}) + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k}) \quad \ldots\text{(i)}$$

and $\vec{r} = (7\hat{i} - 6\hat{j} - 6\hat{k}) + \mu(\hat{i} + 2\hat{j} + 2\hat{k}) \ldots\text{(ii)}$ **(1)**

On comparing Eqs. (i) and (ii) with vector form of equation of line, i.e.

$\vec{r} = \vec{a} + \lambda \vec{b}$, we get

$$\vec{a_1} = 2\hat{i} - 5\hat{j} + \hat{k}, \vec{b_1} = 3\hat{i} + 2\hat{j} + 6\hat{k}$$

and $\vec{a_2} = 7\hat{i} - 6\hat{j} - 6\hat{k}, \vec{b_2} = \hat{i} + 2\hat{j} + 2\hat{k}$ **(1)**

We know that, angle between two lines is given by

$$\cos\theta = \frac{|\vec{b_1} \cdot \vec{b_2}|}{|\vec{b_1}||\vec{b_2}|}$$

$$\therefore \cos\theta = \left|\frac{(3\hat{i} + 2\hat{j} + 6\hat{k}) \cdot (\hat{i} + 2\hat{j} + 2\hat{k})}{\sqrt{(3)^2 + (2)^2 + (6)^2} \cdot \sqrt{(1)^2 + (2)^2 + (2)^2}}\right|$$

(1)

$$\Rightarrow \quad \cos\theta = \left|\frac{3 + 4 + 12}{\sqrt{49} \times \sqrt{9}}\right|$$

$$\Rightarrow \quad \cos\theta = \left|\frac{19}{7 \times 3}\right| \Rightarrow \cos\theta = \frac{19}{21}$$

Hence, angle between given two lines is

$\theta = \cos^{-1}\left(\frac{19}{21}\right)$. **(1)**

21. Show that the lines $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}$ and $\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}$ intersect. Also, find their point of intersection. **Delhi 2014**

The given lines are

$$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}=\lambda \quad \text{[let] ...(i)}$$

and $$\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}=\mu \quad \text{[let] ...(ii)}$$

Then, any point on line (i) is

$$P(3\lambda-1, 5\lambda-3, 7\lambda-5) \quad \text{...(iii)}$$

and any point on line (ii) is

$$Q(\mu+2, 3\mu+4, 5\mu+6) \quad \text{...(iv)}$$

If lines (i) and (ii) intersect, then these points must coincide.

$$\therefore \quad 3\lambda-1=\mu+2$$
$$5\lambda-3=3\mu+4$$
$$7\lambda-5=5\mu+6$$
$$\Rightarrow \quad 3\lambda-\mu=3 \quad \text{...(v)}$$
$$5\lambda-3\mu=7 \quad \text{...(vi)}$$
$$7\lambda-5\mu=11 \quad \text{...(vii)}$$ **(1)**

On multiplying Eq. (v) by 3 and then subtracting Eq. (vi) from it, we get

$$9\lambda-3\mu-5\lambda+3\mu=9-7$$
$$\Rightarrow \quad 4\lambda=2 \Rightarrow \lambda=\frac{1}{2}$$

On putting the value of λ in Eq. (v), we get

$$3\times\frac{1}{2}-\mu=3$$
$$\Rightarrow \quad \frac{3}{2}-\mu=3 \Rightarrow \mu=-\frac{3}{2}$$ **(1)**

On putting the values of λ and μ in Eq. (vii)

On putting the values of λ and μ in Eq. (vii), we get

$$7 \times \frac{1}{2} - 5\left(-\frac{3}{2}\right) = 11$$

$\Rightarrow \quad \frac{7}{2} + \frac{15}{2} = 11 \Rightarrow \frac{22}{2} = 11$

$\Rightarrow \quad 11 = 11$, which is true. **(1)**

Hence, lines (i) and (ii) intersect and their point of intersection is

$$P\left(3 \times \frac{1}{2} - 1, 5 \times \frac{1}{2} - 3, 7 \times \frac{1}{2} - 5\right)$$

[put $\lambda = \frac{1}{2}$ in Eq. (iii)]

i.e. $\quad P\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$ **(1)**

22. Find the value of p, so that the lines

$$l_1 : \frac{1-x}{3} = \frac{7y-14}{p} = \frac{z-3}{2}$$

and $\quad l_2 : \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$

are perpendicular to each other. Also, find the equation of a line passing through a point $(3, 2, -4)$ and parallel to line l_1.

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Equation of the given lines can be written in standard form as

$l_1 : \frac{x-1}{-3} = \frac{y-2}{\frac{p}{7}} = \frac{z-3}{2}$

and $l_2 : \frac{x-1}{-\frac{3p}{7}} = \frac{y-5}{1} = \frac{z-6}{-5}$ **(1)**

Direction ratios of these lines are $-3, \frac{p}{7}, 2$

3p

and $-\frac{3p}{7}, 1, -5$, respectively. **(1)**

We know that, two lines of direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular to each other, if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$\therefore \quad (-3)\left(\frac{-3p}{7}\right) + \left(\frac{p}{7}\right)(1) + (2)(-5) = 0$$

$$\Rightarrow \quad \frac{9p}{7} + \frac{p}{7} - 10 = 0$$

$$\Rightarrow \quad \frac{10p}{7} = 10 \Rightarrow p = 7$$ **(1)**

Thus, the value of p is 7.

Also, we know that, the equation of a line which passes through the point (x_1, y_1, z_1) with direction ratios a, b, c is given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Hence, required line is parallel to line l_1.

So, $a = -3,\ b = \frac{7}{7} = 1$ and $c = 2$

Now, equation of line passing through the point $(3, 2, -4)$ and having direction ratios $(-3, 1, 2)$ is

$$\frac{x - 3}{-3} = \frac{y - 2}{1} = \frac{z + 4}{2}$$

$$\Rightarrow \quad \frac{3 - x}{3} = \frac{y - 2}{1} = \frac{z + 4}{2}$$ **(1)**

23. A line passes through the point (2, – 1, 3) and is perpendicular to the lines

$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda\,(2\hat{i} - 2\hat{j} + \hat{k})$

and $\vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + \mu\,(\hat{i} + 2\hat{j} + 2\hat{k})$. Obtain its equation in vector and cartesian forms.

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Given lines are $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$

and $\vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$.

On comparing with vector form of equation of line $\vec{r} = a + \lambda b$, we get

$b_1 = 2\hat{i} - 2\hat{j} + \hat{k}$ and $b_2 = \hat{i} + 2\hat{j} + 2\hat{k}$. The required line is perpendicular to the given lines. **(1)**

So, it is parallel to the vector

$$\vec{b} = \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= (-4-2)\,\hat{i} - (4-1)\,\hat{j} + (4+2)\,\hat{k}$$

$$= -6\hat{i} - 3\hat{j} + 6\hat{k}$$ **(1)**

Thus, the required line passes through the point (2, – 1, 3) and parallel to the vector $-6\hat{i} - 3\hat{j} + 6\hat{k}$.

So, its vector equation is

$$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda\,(-6\hat{i} - 3\hat{j} + 6\hat{k})$$

The equation can be rewritten as

$$x\hat{i} + y\hat{j} + z\hat{k} = (2-6\lambda)\,\hat{i} + (-1-3\lambda)\,\hat{j} + (3+6\lambda)\,\hat{k}$$

(1)

On comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ from both sides, we get

$$x = 2 - 6\lambda, \quad y = -1 - 3\lambda, \quad z = 3 + 6\lambda$$

$$\Rightarrow \frac{x-2}{-6} = \lambda, \quad \frac{y+1}{-3} = \lambda, \quad \frac{z-3}{6} = \lambda$$

$$\Rightarrow \frac{2-x}{6} = \frac{-y-1}{3} = \frac{z-3}{6}$$

which is the required cartesian form of the line. **(1)**

24. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$$

and $\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$.

Foreign 2014; Delhi 2008

Given equations of lines are

$$\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k}) \quad \text{...(i)}$$

and $$\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu(3\hat{i} - 5\hat{j} + 2\hat{k}) \text{...(ii)}$$

On comparing above equations with vector equation

$$\vec{r} = \vec{a} + \lambda\vec{b}, \text{ we get}$$

$$\vec{a_1} = \hat{i} + \hat{j}, \vec{b_1} = 2\hat{i} - \hat{j} + \hat{k}$$

and $$\vec{a_2} = 2\hat{i} + \hat{j} - \hat{k}, \vec{b_2} = 3\hat{i} - 5\hat{j} + 2\hat{k}$$ **(1)**

Now, we know that, the shortest distance between two lines is given by

$$d = \left| \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right| \quad \text{...(iii)}$$

$$\therefore \quad \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix}$$

$$= \hat{i}(-2+5) - \hat{j}(4-3) + \hat{k}(-10+3)$$

$$\Rightarrow \vec{b_1} \times \vec{b_2} = 3\hat{i} - \hat{j} - 7\hat{k} \quad \text{...(iv)} \ \mathbf{(1)}$$

and $$|\vec{b_1} \times \vec{b_2}| = \sqrt{(3)^2 + (-1)^2 + (-7)^2}$$

$$= \sqrt{9 + 1 + 49} = \sqrt{59} \quad \text{...(v)}$$

Also, $$\vec{a_2} - \vec{a_1} = (2\hat{i} + \hat{j} - \hat{k}) - (\hat{i} + \hat{j})$$

$$= \hat{i} - \hat{k} \quad \text{...(vi)} \ \mathbf{(1)}$$

From Eqs. (iii), (iv), (v) and (vi), we get

$$d = \left| \frac{(3\hat{i} - \hat{j} - 7\hat{k}) \cdot (\hat{i} - \hat{k})}{\sqrt{59}} \right|$$

$$\Rightarrow \quad d = \left| \frac{3 - 0 + 7}{\sqrt{59}} \right| = \frac{10}{\sqrt{59}}$$

Hence, required shortest distance is $\frac{10}{\sqrt{59}}$ units. **(1)**

25. Find the shortest distance between the two lines whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) \text{ and}$$

$$\vec{r} = (4\hat{i} + 5\hat{j} + \hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$$ **Delhi 2014C**

Given equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) \quad \ldots(i)$$

$$\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k}) \quad \ldots(ii)$$

On comparing with $\vec{r} = \vec{a} + \lambda\vec{b}$, we get

$$\vec{a_1} = \hat{i} + 2\hat{j} + 3\hat{k}, \vec{b_1} = \hat{i} - 3\hat{j} + 2\hat{k}$$

and $\vec{a_2} = 4\hat{i} + 5\hat{j} + 6\hat{k}, \vec{b_2} = 2\hat{i} + 3\hat{j} + \hat{k}$

Now, $\vec{a_2} - \vec{a_1} = (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$$= 3\hat{i} + 3\hat{j} + 3\hat{k}$$

and $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$ **(1)**

$$= \hat{i}(-3-6) - \hat{j}(1-4) + \hat{k}(3+6) = -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(-9)^2 + 3^2 + 9^2}$

$\Rightarrow \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{81 + 9 + 81} = \sqrt{171}$ **(1)**

Now, $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} - \vec{b_2})$

$$= (3\hat{i} + 3\hat{j} + 3\hat{k}) \cdot (-9\hat{i} + 3\hat{j} + 9\hat{k})$$

$$= -27 + 9 + 27 = 9$$ **(1)**

$\therefore$Shortest distance between two lines is

$$SD = \left| \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right| = \left| \frac{9}{\sqrt{171}} \right| \text{ units}$$ **(1)**

26. Find the shortest distance between the following lines:

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}, \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}.$$

Foreign 2014; Delhi 2008

Given equations of lines are

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1} \quad \text{...(i)}$$

and $$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1} \quad \text{...(ii)}$$

On comparing above equations with one point form of equation of line, i.e.

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}, \text{ we get}$$

$a_1=1, b_1=-2, c_1=1, x_1=3,$
$y_1=5, z_1=7$

and $a_2=7, b_2=-6,$
$c_2=1, x_2=-1,$
$y_2=-1, z_2=-1$ **(1)**

We know that, the shortest distance between two lines is given by

$$d=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

$$\therefore \quad d=\frac{\begin{vmatrix} -4 & -6 & -8 \\ 1 & -2 & 1 \\ 7 & -6 & 1 \end{vmatrix}}{\sqrt{(-2+6)^2+(7-1)^2+(-6+14)^2}} \quad \textbf{(1)}$$

$[\because x_2-x_1=-1-3=-4, y_2-y_1=-1-5=-6,$
$z_2-z_1=-1-7=-8]$

$$=\left|\frac{-4(-2+6)+6(1-7)-8(-6+14)}{\sqrt{(4)^2+(6)^2+(8)^2}}\right|$$

$$=\left|\frac{-4(4)+6(-6)-8(8)}{\sqrt{16+36+64}}\right|=\left|\frac{-16-36-64}{\sqrt{116}}\right|$$

(1)

$$=\left|\frac{-116}{\sqrt{116}}\right|=\frac{116}{\sqrt{116}}=\frac{(\sqrt{116})^2}{\sqrt{116}}=\sqrt{116}$$

Hence, the required shortest distance is $\sqrt{116}$ units. **(1)**

27. Find the distance between the lines l_1 and l_2 given by $l_1: \vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$, $l_2: \vec{r}=3\hat{i}+3\hat{j}-5\hat{k}+\mu(4\hat{i}+6\hat{j}+12\hat{k})$. **Foreign 2014**

Given equation of lines are

$$\vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$$

$$\vec{r}=3\hat{i}+3\hat{j}-5\hat{k}+\mu(4\hat{i}+6\hat{j}+12\hat{k})$$

On comparing with $\vec{r}=\vec{a}+\lambda\vec{b}$, we get

$$a_1=\hat{i}+2\hat{j}-4\hat{k},\ \vec{b_1}=2\hat{i}+3\hat{j}+6\hat{k}$$

and $\vec{a_2}=3\hat{i}+3\hat{j}-5\hat{k},\ \vec{b_2}=4\hat{i}+6\hat{j}+12\hat{k}$. **(1)**

Now, $\vec{a_2}-\vec{a_1}=(3\hat{i}+3\hat{j}-5\hat{k})-(\hat{i}+2\hat{j}-4\hat{k})$

$$=2\hat{i}+\hat{j}-\hat{k}$$

and $\vec{b_1}\times\vec{b_2}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&3&6\\4&6&12\end{vmatrix}$

$=\hat{i}(36-36)-\hat{j}(24-24)+\hat{k}(12-12)=0$ **(1)**

So, both given lines are parallel and

$$\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$$

Then, $\vec{b}\times(\overline{a_2}-\overline{a_1})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&3&6\\2&1&-1\end{vmatrix}$

$$=\hat{i}(-3-6)-\hat{j}(-2-12)+\hat{k}(2-6)$$

$= -9i + 14j - 4k$ **(1)**

Now, required distance between given lines is given by

$$d = \left|\frac{\vec{b} \times (\overline{a_2} - \overline{a_1})}{|\vec{b}|}\right| = \left|\frac{-9\hat{i} + 14\hat{j} - 4\hat{k}}{\sqrt{(2)^2 + (3)^2 + (6)^2}}\right|$$

$$= \frac{\sqrt{81 + 196 - 16}}{\sqrt{4 + 9 + 36}} = \frac{\sqrt{261}}{\sqrt{49}}$$

$$= \frac{\sqrt{261}}{7} \text{ units}$$ **(1)**

28. Find the vector and cartesian equations of the line passing through the point (2, 1, 3) and perpendicular to the lines **All India 2014**

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3} \text{ and } \frac{x}{-3} = \frac{y}{2} = \frac{z}{5}.$$

Any line through the point (2, 1, 3) can be written as

$$\frac{x-2}{a} = \frac{y-1}{b} = \frac{z-3}{c} \quad \text{...(i)}$$

where, a, b and c are the direction ratios of line (i).

Now, the line (i) is perpendicular to the lines

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$$

and $$\frac{x-0}{-3} = \frac{y-0}{2} = \frac{z-0}{5}.$$

Direction ratios of these two lines are (1, 2, 3) and (−3, 2, 5), respectively. **(1)**

$\therefore \quad a + 2b + 3c = 0 \quad$ ...(ii)

[$\because$ if two lines are perpendicular, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$]

and $\quad -3a + 2b + 5c = 0 \quad$ ...(iii)

In Eqs. (ii) and (iii), by cross-multiplication, we get we get

$$\frac{a}{10-6} = \frac{-b}{-9-5} = \frac{c}{2+6}$$

$$\Rightarrow \qquad \frac{a}{4} = \frac{b}{14} = \frac{c}{8}$$

$$\Rightarrow \qquad \frac{a}{2} = \frac{b}{7} = \frac{c}{4} = \lambda \qquad \text{[say]}$$

$\therefore \qquad a = 2\lambda, b = 7\lambda$ and $c = 6\lambda$ **(1)**

On substituting the values of a, b and c in Eq. (i), we get

$$\frac{x-2}{2\lambda} = \frac{y-1}{7\lambda} = \frac{z-3}{6\lambda}$$

$$\Rightarrow \qquad \frac{x-2}{2} = \frac{y-1}{7} = \frac{z-3}{6} \qquad \textbf{(1)}$$

which is the required cartesian equation of the line.

The vector equation of line which passes through (2, 1, 3) and parallel to the vector $2\hat{i} + 7\hat{j} + 6\hat{k}$ is

$$\vec{r} = 2\hat{i} + \hat{j} + 3\hat{k} + \lambda(2\hat{i} + 7\hat{j} + 6\hat{k})$$

which is the required vector equation of the line. **(1)**

29. The cartesian equation of a line is $6x - 2 = 3y + 1 = 2z - 2$. Find the direction cosines of the line. Write down the cartesian and vector equations of a line passing through $(2, -1, -1)$ which are parallel to the given line. **Delhi 2013C**

Given equation of line is

$$6x - 2 = 3y + 1 = 2z - 2$$

or $$\frac{x - 2/6}{1/6} = \frac{y + 1/3}{1/3} = \frac{z - 2/2}{1/2}$$

$$\Rightarrow \quad \frac{x-1/3}{1/6}=\frac{y+1/3}{1/3}=\frac{z-1}{1/2}$$

$\therefore$ DC's of a line are $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}$. **(1)**

The equation of a line passing through $(2, -1, -1)$ and parallel to the given line is

$$\frac{x-2}{1/6}=\frac{y+1}{1/3}=\frac{z+1}{1/2}=\lambda$$ [say]**(1)**

$$\left[\because \frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}\right]$$

$$\Rightarrow \quad x=2+\frac{\lambda}{6}, y=-1+\frac{\lambda}{3} \text{ and } z=-1+\frac{\lambda}{2}$$

Now, $x\hat{i}+y\hat{j}+z\hat{k}=\left(2+\frac{\lambda}{6}\right)\hat{i}+\left(-1+\frac{\lambda}{3}\right)\hat{j}+\left(-1+\frac{\lambda}{2}\right)\hat{k}$ **(1)**

$$\Rightarrow \quad \vec{r}=(2\hat{i}-\hat{j}-\hat{k})+\lambda\left(\frac{1}{6}\hat{i}+\frac{1}{3}\hat{j}+\frac{1}{2}\hat{k}\right)$$

which is the required equation of line in vector form. **(1)**

30. Find the shortest distance between the two lines whose vector equations are

$$\vec{r}=(6\hat{i}+2\hat{j}+2\hat{k})+\lambda(\hat{i}-2\hat{j}+2\hat{k})$$

and $\vec{r}=(-4\hat{i}-\hat{k})+\mu(3\hat{i}-2\hat{j}-2\hat{k})$.

Delhi 2013C; Foreign 2011

Given equations of lines are

$$\vec{r} = (6\hat{i} + 2\hat{j} + 2\hat{k}) + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$$

and $\vec{r} = (-4\hat{i} - \hat{k}) + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$

which are of the form $\vec{r} = \vec{a} + \lambda\vec{b}$.

Here, $\vec{a_1} = 6\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b_1} = \hat{i} - 2\hat{j} + 2\hat{k}$;

$$\vec{a_2} = -4\hat{i} - \hat{k} \text{ and } \vec{b_2} = 3\hat{i} - 2\hat{j} - 2\hat{k}$$

Then, $\vec{a_2} - \vec{a_1} = (-4\hat{i} - \hat{k}) - (6\hat{i} + 2\hat{j} + 2\hat{k})$

$$= -10\hat{i} - 2\hat{j} - 3\hat{k} \quad \textbf{(1)}$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix}$$

$$= \hat{i}(4 + 4) - \hat{j}(-2 - 6) + \hat{k}(-2 + 6)$$

$$= 8\hat{i} + 8\hat{j} + 4\hat{k} \quad \textbf{(1)}$$

$$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(8)^2 + (8)^2 + (4)^2}$$

$$= \sqrt{64 + 64 + 16} = \sqrt{144} = 12 \quad \textbf{(1)}$$

Now, $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})$

$$= (-10\hat{i} - 2\hat{j} - 3\hat{k}) \cdot (8\hat{i} + 8\hat{j} + 4\hat{k})$$

$$= -80 - 16 - 12 = -108$$

$$\therefore \quad \text{Required SD} = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

$$= \left| \frac{-108}{12} \right| = 9 \text{ units} \quad \textbf{(1)}$$

31. Show that the lines

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k});$$

$$\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$$

are intersecting. Hence, find their point of intersection. **All India 2013**

Given vector lines are

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$$

and $\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$

Their cartesian form are

$$\frac{x-3}{1} = \frac{y-2}{2} = \frac{z+4}{2} = r \quad \text{[say] ...(i)}$$

and $\frac{x-5}{3} = \frac{y+2}{2} = \frac{z-0}{6} = p$ [say] ...(ii)**(1)**

Let $(r+3, 2r+2, 2r-4)$ and $(3p+5, 2p-2, 6p)$ be two points on the lines (i) and (ii), respectively.

If these lines intersect each other, then

$$r + 3 = 3p + 5$$

$\Rightarrow \quad r - 3p = 2$ ...(iii)

$$2r + 2 = 2p - 2$$

$\Rightarrow \quad r - p = -2$ ...(iv)

and $2r - 4 = 6p \Rightarrow 2r - 6p = 4$

$\Rightarrow \quad r - 3p = 2$ ...(v) **(1)**

Now, subtracting Eq. (v) from Eq. (iv), we get

$$2p = -4 \Rightarrow p = -2$$

On putting $p = -2$ in Eq. (iv), we get

$$r - (-2) = -2 \Rightarrow r = -4$$

$\therefore$ Any point on line (i) is

$(-4+3, -8+2, -8-4) = (-1, -6, -12)$ **(1)**

and any point on line (ii) is

$(-6+5, -4-2, -12) = (-1, -6, -12)$

Since, both points are same, therefore both lines intersect each other at point $(-1, -6, -12)$. **(1)**

32. Find the vector and cartesian equations of line passing through point (1, 2, – 4) and perpendicular to two lines

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$$

and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$. **Delhi 2012**

Let the required equation of line passing through (1,2,–4) be

$$\frac{x-1}{a}=\frac{y-2}{b}=\frac{z+4}{c} \quad \text{...(i)}$$

Given that line (i) is perpendicular to lines

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \quad \text{...(ii)}$$

and $$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5} \quad \text{...(iii)}$$ **(1)**

We know that, when two lines are perpendicular, then we have
$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$, where a_1, b_1, c_1 and a_2, b_2, c_2 are the DR's of two lines.

Using this property, first in Eqs. (i) and (ii) and then in Eqs. (i) and (iii), we get

$$3a - 16b + 7c = 0 \quad \text{...(iv)}$$

and $$3a + 8b - 5c = 0 \quad \text{...(v)}$$ **(1)**

On subtracting Eq. (v) from Eq. (iv), we get

$$3a - 16b = -7c$$
$$-3a - 8b = -5c$$
$$-24b = -12c$$

$$\Rightarrow \quad b = \frac{c}{2}$$

On putting $b = \frac{c}{2}$ in Eq. (iv), we get

$$3a - 16\left(\frac{c}{2}\right) + 7c = 0$$

$$\Rightarrow \quad 3a - 8c + 7c = 0$$

$$\Rightarrow \quad 3a - c = 0$$

$$\Rightarrow \quad a = \frac{c}{3} \quad \textbf{(1)}$$

On putting $a = \frac{c}{3}$ and $b = \frac{c}{2}$ in Eq. (i), we get the required equation of line in cartesian form as

$$\frac{x-1}{\left(\frac{c}{3}\right)} = \frac{y-2}{\left(\frac{c}{2}\right)} = \frac{z+4}{c}$$

[on multiplying denominator by 6]

$$\Rightarrow \quad \frac{x-1}{2c} = \frac{y-2}{3c} = \frac{z+4}{6c}$$

$$\Rightarrow \quad \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}.$$

[dividing denominator by c]

Also, the vector equation of line is

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}) \quad \textbf{(1)}$$

33. Find the angle between following pair of lines

$$\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3}$$

and $\frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4}$

and check whether the lines are parallel or perpendicular. **HOTS; Delhi 2011**

Firstly, we convert the given lines in standard form and then use the relation

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}},$$

to find the angle between them.

Given equations of two lines are

$$\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3}$$

and $\frac{x+2}{-1}=\frac{2y-8}{4}=\frac{z-5}{4}$

Above equations can be written as

$$\frac{x-2}{2}=\frac{y-1}{7}=\frac{z+3}{-3} \quad \text{...(i)}$$

and $\frac{x+2}{-1}=\frac{y-4}{2}=\frac{z-5}{4}$ ...(ii) **(1)**

On comparing Eqs. (i) and (ii) with one point form of equation of line

$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$, we get

$$a_1=2, b_1=7, c_1=-3$$

and $a_2=-1, b_2=2, c_2=4$

We know that, angle between two lines is given by

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}} \quad \textbf{(1)}$$

$$\therefore \cos\theta=\frac{(2)(-1)+(7)(2)+(-3)(4)}{\sqrt{(2)^2+(7)^2+(-3)^2}\cdot\sqrt{(-1)^2+(2)^2+(4)^2}}$$

$$\therefore \cos\theta=\frac{-2+14-12}{\sqrt{62}\times\sqrt{21}}=\frac{0}{\sqrt{62}\times\sqrt{21}}=0 \quad \textbf{(1)}$$

$$\Rightarrow \cos\theta=\cos\frac{\pi}{2} \quad \left[\because 0=\cos\frac{\pi}{2}\right]$$

$$\Rightarrow \theta=\frac{\pi}{2}$$

Hence, angle between them is $\frac{\pi}{2}$. Since, angle between the two lines is $\frac{\pi}{2}$, therefore the given pair of lines are perpendicular to each other. **(1)**

NOTE Please be careful while taking DR's of a line, the line should be in symmetrical or in standard form, otherwise there may be chances of error.

34. Find the shortest distance between lines whose vector equations are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$$

$$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}.$$

HOTS; All India 2011

Firstly, convert both the equations in the vector form which is $\vec{r} = \vec{a} + \lambda\vec{b}$. Then, apply the shortest distance formula,

i.e. $$d = \left| \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

Given equations are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} \quad \text{...(i)}$$

and $\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$ ...(ii)

Firstly, we convert both equations in the vector form as $\vec{r} = \vec{a} + \lambda\vec{b}$ ...(iii)

So, Eq. (i) can be written as

$$\vec{r} = \hat{i} - t\hat{i} + t\hat{j} - 2\hat{j} + 3\hat{k} - 2t\hat{k}$$

$\Rightarrow$ $\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k})$ ...(iv) **(1)**

and Eq. (ii) can be written as

$$\vec{r} = s\hat{i} + \hat{i} + 2s\hat{j} - \hat{j} - 2s\hat{k} - \hat{k}$$

$\Rightarrow$ $\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k})$ ...(v)

Now, from Eqs. (iii), (iv) and (v), we get

$$\vec{a_1} = \hat{i} - 2\hat{j} + 3\hat{k}, \ \vec{b_1} = -\hat{i} + \hat{j} - 2\hat{k}$$

$$\vec{a_2} = \hat{i} - \hat{j} - \hat{k}, \ \vec{b_2} = \hat{i} + 2\hat{j} - 2\hat{k}$$

$a_2 = \hat{i} - \hat{j} - \hat{k}, b_2 = \hat{i} + 2\hat{j} - 2\hat{k}$

Then, $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$

$$= \hat{i}(-2+4) - \hat{j}(2+2) + \hat{k}(-2-1)$$

$\Rightarrow \vec{b_1} \times \vec{b_2} = 2\hat{i} - 4\hat{j} - 3\hat{k}$

$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(2)^2 + (-4)^2 + (-3)^2}$

$= \sqrt{4+16+9} = \sqrt{29}$ **(1)**

Also, $\vec{a_2} - \vec{a_1} = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k})$

$= \hat{j} - 4\hat{k}$

We know that, the shortest distance between the lines is given as

$$d = \left| \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|$$ **(1)**

Hence, required shortest distance,

$$d = \left| \frac{(2\hat{i} - 4\hat{j} - 3\hat{k}) \cdot (\hat{j} - 4\hat{k})}{\sqrt{29}} \right|$$

$$= \left| \frac{0 - 4 + 12}{\sqrt{29}} \right| = \frac{8}{\sqrt{29}}$$

$\Rightarrow \quad d = \frac{8\sqrt{29}}{29}$ units **(1)**

35. Find shortest distance between the lines $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$.

Foreign 2011; All India 2...

Do same as Que. 34. $\left[\textbf{Ans.} \ \frac{3\sqrt{2}}{2} \text{ units} \right]$

36. Find the equation of the perpendicular from point (3, –1, 11) to line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$.
Also, find the coordinates of foot of perpendicular and the length of perpendicular. **HOTS; All India 2011C**

> Firstly, determine any point P on the given line and DR's between given point Q and P, using the relation $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$, where (a_1, b_1, c_1) and (a_2, b_2, c_2) are DR's of PQ and given line.

Given equation of line AB is

$$\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda \quad \text{[say]}$$

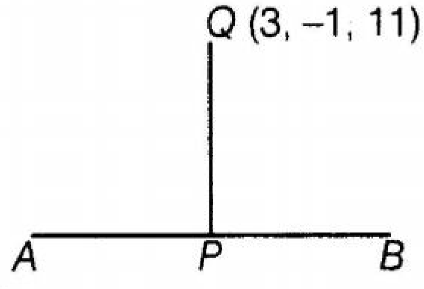
$$\Rightarrow \quad \frac{x}{2} = \lambda, \frac{y-2}{3} = \lambda \text{ and } \frac{z-3}{4} = \lambda$$

$$\Rightarrow \quad x = 2\lambda, y = 3\lambda + 2$$

and $\quad z = 4\lambda + 3 \quad$ **(1)**

$\therefore$ Any point P on the given line

$$= (2\lambda, 3\lambda + 2, 4\lambda + 3)$$



Let P be the foot of perpendicular drawn from point $Q(3, -1, 11)$ on line AB. Now, DR's of line

$QP = (2\lambda - 3, 3\lambda + 2 + 1, 4\lambda + 3 - 11)$ **(1)**

$\Rightarrow$ DR's of line $QP = (2\lambda - 3, 3\lambda + 3, 4\lambda - 8)$

Here, $\quad a_1 = 2\lambda - 3, b_1 = 3\lambda + 3, c_1 = 4\lambda - 8,$

and $\quad a_2 = 2, b_2 = 3, c_2 = 4$

Since, $\quad QP \perp AB$

$\therefore$ We have, $a_1a_2 + b_1b_2 + c_1c_2 = 0 \quad$ ...(i)

$$\Rightarrow \quad 2(2\lambda - 3) + 3(3\lambda + 3) + 4(4\lambda - 8) = 0$$
$$\Rightarrow \quad 4\lambda - 6 + 9\lambda + 9 + 16\lambda - 32 = 0$$
$$\Rightarrow \quad 29\lambda - 29 = 0$$
$$\Rightarrow \quad 29\lambda = 29 \Rightarrow \lambda = 1$$ **(1)**

$\therefore$ Foot of perpendicular $P = (2, 3 + 2, 4 + 3)$
$= (2, 5, 7)$

Now, equation of perpendicular QP, where $Q(3, -1, 11)$ and $P(2, 5, 7)$, is

$$\frac{x-3}{2-3} = \frac{y+1}{5+1} = \frac{z-11}{7-11}$$

$$\left[\begin{array}{l}\text{using two points form of equation of line,} \\ \text{i.e. } \dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1}\end{array}\right]$$

$$\Rightarrow \quad \frac{x-3}{-1} = \frac{y+1}{6} = \frac{z-11}{-4}$$

Now, length of perpendicular QP = distance between points $Q(3, -1, 11)$ and $P(2, 5, 7)$

$$= \sqrt{(2-3)^2 + (5+1)^2 + (7-11)^2}$$

$$\left[\begin{array}{l}\because \text{Distance} \\ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}\end{array}\right]$$

$$= \sqrt{1 + 36 + 16} = \sqrt{53}$$

Hence, length of perpendicular is $\sqrt{53}$. **(1)**

37. Find the perpendicular distance of point (1, 0, 0) from the lines $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$. Also, find the coordinates of foot of perpendicular and equation of perpendicular. **Delhi 2011C**

Do same as Que. 36.

[**Ans.** Length of perpendicular is $\sqrt{53}$.

Coordinates of Food of perpendicular

$$= (3, \ -4, \ -2)$$

$\therefore$ Equation of perpendicular $= \dfrac{x-1}{2} = \dfrac{y}{-4} = \dfrac{z}{-2}$]

38. Find the points on the line $\dfrac{x+2}{3} = \dfrac{y+1}{2} = \dfrac{z-3}{2}$ at a distance of 5 units from the point $P(1,3,3)$. **All India 2010**

Given equation of line is

$$\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2} = \lambda \quad \text{[say]}$$

$$\Rightarrow \quad \frac{x+2}{3} = \lambda, \frac{y+1}{2} = \lambda, \frac{z-3}{2} = \lambda$$

$$\Rightarrow \quad x = 3\lambda - 2, y = 2\lambda - 1, z = 2\lambda + 3$$

So, we have the point

$$Q\,(3\lambda - 2, 2\lambda - 1, 2\lambda + 3) \quad \text{...(i)} \ \textbf{(1)}$$

Now, given that distance between two points $P\,(1, 3, 3)$ and $Q\ (3\lambda - 2, \ 2\lambda - 1, 2\lambda + 3)$ is 5 units, i.e. $PQ = 5$

$$\Rightarrow \quad \sqrt{\begin{bmatrix}(3\lambda - 2 - 1)^2 + (2\lambda - 1 - 3)^2 \\ + (2\lambda + 3 - 3)^2\end{bmatrix}} = 5$$

$$\left[\because \text{distance} = \sqrt{\begin{matrix}(x_2 - x_1)^2 + (y_2 - y_1)^2 \\ + (z_2 - z_1)^2\end{matrix}}\right]$$

$$\Rightarrow \sqrt{(3\lambda - 3)^2 + (2\lambda - 4)^2 + (2\lambda)^2} = 5 \quad \textbf{(1)}$$

On squaring both sides, we get

$$(3\lambda - 3)^2 + (2\lambda - 4)^2 + (2\lambda)^2 = 25$$

$$\Rightarrow \quad 9\lambda^2 + 9 - 18\lambda + 4\lambda^2 + 16 - 16\lambda + 4\lambda^2 = 25$$

$$\Rightarrow 17\lambda^2 - 34\lambda = 0 \Rightarrow 17\lambda(\lambda - 2) = 0 \qquad \textbf{(1)}$$

$$\Rightarrow \text{ Either } 17\lambda = 0 \text{ or } \lambda - 2 = 0$$

$$\therefore \quad \lambda = 0 \text{ or } 2$$

On putting $\lambda = 0$ and $\lambda = 2$ in Eq. (i), we get the required point as $(-2, -1, 3)$ or $(4, 3, 7)$. **(1)**

39. Find the shortest distance between the lines

$$l_1 : \frac{x-1}{1} = \frac{y-2}{-1} = \frac{z-1}{1}$$

$$l_2 : \frac{x-2}{2} = \frac{y+1}{1} = \frac{z+1}{2}.$$

All India 2009C

Do same as Que. 25. $\left[\textbf{Ans. } \frac{3}{\sqrt{2}} \text{ units}\right]$

40. Find shortest distance between lines

$$\vec{r} = (1 + 2\lambda)\hat{i} + (1 - \lambda)\hat{j} + \lambda\hat{k}$$

$$\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k}).$$

All India 2009

Do same as Que. 34. $\left[\textbf{Ans. } \frac{3}{\sqrt{29}} \text{ units}\right]$

41. Find the value of λ, so that following lines are perpendicular to each other

$$\frac{x+5}{5\lambda+2} = \frac{2-y}{5} = \frac{1-z}{-1} \text{ and } \frac{x}{1} = \frac{2y+1}{4\lambda} = \frac{1-z}{-3}.$$

Delhi 2009

Firstly, convert the given equations of lines into one point form of the line, which is of form $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ and then use the condition $a_1a_2+b_1b_2+c_1c_2=0$ for perpendicularity of two lines and get value of λ.

Given equation of lines are

$$\frac{x+5}{5\lambda+2}=\frac{2-y}{5}=\frac{1-z}{-1}$$

and $$\frac{x}{1}=\frac{2y+1}{4\lambda}=\frac{1-z}{-3}$$

Above equations can be written as

$$\frac{x+5}{5\lambda+2}=\frac{y-2}{-5}=\frac{z-1}{1} \quad \ldots(i)$$

and $$\frac{x}{1}=\frac{2\left(y+\frac{1}{2}\right)}{4\lambda}=\frac{z-1}{3}$$

$\Rightarrow$ $$\frac{x}{1}=\frac{y+\frac{1}{2}}{2\lambda}=\frac{z-1}{3} \quad \ldots(ii)$$ **(1)**

On comparing Eqs. (i) and (ii) with one point form of line

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}, \text{ we get}$$

$$a_1=5\lambda+2,\ b_1=-5,\ c_1=1$$

and $$a_2=1,\ b_2=2\lambda,\ c_2=3$$ **(1)**

Since, the two lines are perpendicular.

$\therefore$ $$a_1a_2+b_1b_2+c_1c_2=0$$

$\Rightarrow$ $$1(5\lambda+2)+2\lambda(-5)+3(1)=0$$

$$\Rightarrow \quad 5\lambda + 2 - 10\lambda + 3 = 0 \quad \textbf{(1)}$$
$$\Rightarrow \quad -5\lambda + 5 = 0$$
$$\Rightarrow \quad 5\lambda = 5$$
$$\therefore \quad \lambda = 1 \quad \textbf{(1)}$$

42. Find the value of λ, so that lines

$$\frac{1-x}{3} = \frac{y-2}{2\lambda} = \frac{z-3}{2} \text{ and } \frac{x+1}{3\lambda} = \frac{y-1}{1} = \frac{6-z}{7}$$

are perpendicular to each other. **Delhi 2009**

Do same as Que. 41. **[Ans.** $\lambda = -2$**]**

43. Find the value of λ, so that lines

$$\frac{1-x}{3} = \frac{7y-14}{2\lambda} = \frac{5z-10}{11}$$

and $$\frac{7-7x}{3\lambda} = \frac{y-5}{1} = \frac{6-z}{5}$$

are perpendicular to each other. **Delhi 2009**

Do same as Que. 41. **[Ans.** $\lambda = 7$**]**

44. Find the length and foot of perpendicular drawn from the point (2, –1, 5) to line

$$\frac{x-11}{10} = \frac{y+2}{-4} = \frac{z+8}{-11}.$$ **All India 2008**

Do same as Que. 36.

[Ans. Length = $\sqrt{14}$ units and foot of prependicular = (1, 2, 3)**]**

6 Marks Questions

45. Find the distance of the point $P(-1, -5, -10)$ from the point of intersection of the line joining the points $A(2, -1, 2)$ and $B(5, 3, 4)$ with the plane $x - y + z = 5$. **Foreign 2014**

The equation of the line passing through the points $A(2, -1, 2)$ and $B(5, 3, 4)$ is given by

$$\frac{x-2}{5-2} = \frac{y+1}{3+1} = \frac{z-2}{4-2}$$

$$\Rightarrow \quad \frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} = \lambda \quad \text{[say]} \quad \textbf{(1)}$$

$\Rightarrow \quad x = 3\lambda + 2, y = 4\lambda - 1, z = 2\lambda + 2$ **(1)**

Now, putting the values of x, y and z in the equation of the plane $x - y + z = 5$, we get

$$(3\lambda + 2) - (4\lambda - 1) + (2\lambda + 2) = 5 \quad \textbf{(1)}$$

$$\Rightarrow \quad \lambda + 5 = 5$$

$$\therefore \quad \lambda = 0 \quad \textbf{(1)}$$

So, the point of intersection of the line and the plane is $(2, -1, 2)$. **(1)**

$\therefore$ The distance of the point $P(-1, -5, -10)$ and the point of intersection $(2, -1, 2)$ is

$$= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2}$$

$$= \sqrt{3^2 + 4^2 + 12^2} = 13 \text{ units} \quad \textbf{(1)}$$

46. Find the vector and cartesian forms of the equation of the plane passing through the point $(1, 2, -4)$ and parallel to the lines

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

and $\vec{r} = \hat{i} - 3\hat{j} + 5\hat{k} + \mu(\hat{i} + \hat{j} - \hat{k})$. Also, find the distance of the point $(9, -8, -10)$ from the plane thus obtained. Delhi 2014C

Let equation of plane through (1, 2, – 4) be

$$a(x-1)+b(y+2)-c(z+4)=0 \quad \text{...(i)}$$

Given lines are

$$\vec{r}=\hat{i}+2\hat{j}-\hat{k}+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$$

and $\vec{r}=\hat{i}-3\hat{j}+5\hat{k}+\lambda(\hat{i}+\hat{j}-\hat{k})$ **(1)**

The plane (i) is parallel to the given lines,

So, $2a+3b+6c=0$ and $a+b-c=0$ **(1)**

For solving these two equations by cross-multiplication, we get

$$\frac{a}{-3-6}=\frac{-b}{-2-6}=\frac{c}{2-3}$$

$$\Rightarrow \quad \frac{a}{-9}=\frac{b}{8}=\frac{c}{-1}=\lambda \quad \text{[say]}$$

$$\therefore \quad a=-9\lambda,\ b=8\lambda,\ c=-\lambda$$

On putting values of a, b and c in Eq. (i), we get $-9\lambda(x-1)+8\lambda(y-2)-\lambda(z+4)=0$

$\therefore$ Equation of plane in cartesian form is

$$-9\lambda(x-1)+8\lambda(y-2)-\lambda(z+4)=0$$

$$\Rightarrow \quad -9x+9+8y-16-z-4=0$$

$$\Rightarrow \quad 9x-8y+z+11=0 \quad \textbf{(1)}$$

Now, vector form of plane is

$$\vec{r}(9\hat{i}-8\hat{j}+\hat{k})=-11 \quad \textbf{(1)}$$

Also, distance of $(9, -8, -10)$ from the above plane

$$=\left|\frac{9-8(-8)+1(-10)+11}{\sqrt{9^2+(-8)^2+1^2}}\right|$$

$$=\left|\frac{72+64-10+11}{\sqrt{81+64+1}}\right|$$

$$\left[\because D=\left|\frac{Ax+by+Cz+D}{\sqrt{A^2+B^2+C^2}}\right|\right]$$

$$=\left|\frac{146}{\sqrt{146}}\right|=\sqrt{146} \text{ units} \quad \textbf{(1)}$$

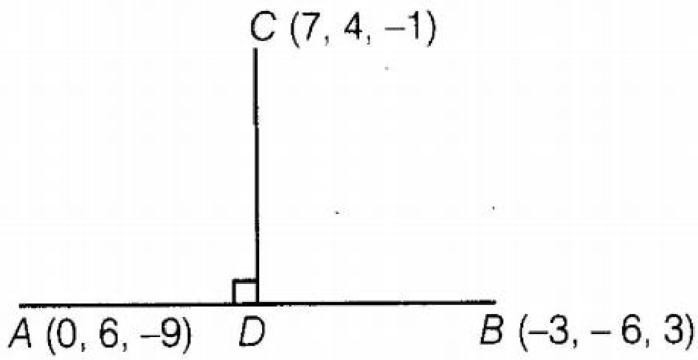
47. Find the equation of line passing through points $A(0, 6, -9)$ and $B(-3, -6, 3)$. If D is the foot of perpendicular drawn from the point $C(7, 4, -1)$ on the line AB, then find the coordinates of point D and equation of line CD. All India 2010C

We know that, equation of line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \quad \ldots(i) \; \textbf{(1)}$$

Here, $A(x_1, y_1, z_1) = (0, 6, -9)$

and $(x_2, y_2, z_2) = (-3, -6, 3)$



$\therefore$ Equation of line AB is given by

$$\frac{x - 0}{-3 - 0} = \frac{y - 6}{-6 - 6} = \frac{z + 9}{3 + 9}$$

$$\Rightarrow \quad \frac{x}{-3} = \frac{y - 6}{-12} = \frac{z + 9}{12}$$

$$\Rightarrow \quad \frac{x}{-1} = \frac{y - 6}{-4} = \frac{z + 9}{4} \quad \textbf{(1)}$$

[dividing denominator by 3]

Next, we have to find coordinates of foot of perpendicular D.

Now, let $\frac{x}{-1} = \frac{y - 6}{-4} = \frac{z + 9}{4} = \lambda$ [say]

$\Rightarrow$ $x = -\lambda$

$y - 6 = -4\lambda$ and $z + 9 = 4\lambda$

$\Rightarrow$ $x = -\lambda$

$y = -4\lambda + 6$

and $z = 4\lambda - 9$ **(1)**

Let coordinates of

$$D = (-\lambda, -4\lambda + 6, 4\lambda - 9) \quad \ldots(ii)$$

Now, DR's of line CD are

$$(-\lambda - 7, -4\lambda + 6 - 4, 4\lambda - 9 + 1)$$
$$= (-\lambda - 7, -4\lambda + 2, 4\lambda - 8)$$

Now, $CD \perp AB$

$\therefore \quad a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$ **(1)**

where, $a_1 = -\lambda - 7, b_1 = -4\lambda + 2,$

$c_1 = 4\lambda - 8$ [DR's of line CD]

and $a_2 = -1, b_2 = -4, c_2 = 4$

[DR's of line AB]

$\Rightarrow -1(-\lambda - 7) - 4(-4\lambda + 2) + 4(4\lambda - 8) = 0$

$\Rightarrow \lambda + 7 + 16\lambda - 8 + 16\lambda - 32 = 0$

$\Rightarrow 33\lambda - 33 = 0$

$\Rightarrow 33\lambda = 33$

$\therefore \lambda = 1$ **(1)**

On putting $\lambda = 1$ in Eq. (ii), we get required foot of perpendicular,

$$D = (-1, 2, -5)$$

Also, we have to find equation of line CD, where, C (7, 4, – 1) and D (–1, 2, – 5).

$\therefore$ Required equation of line is

$$\frac{x-7}{-1-7} = \frac{y-4}{2-4} = \frac{z+1}{-5+1} \text{ [using Eq. (i)]}$$

$$\Rightarrow \frac{x-7}{-8} = \frac{y-4}{-2} = \frac{z+1}{-4}$$

$$\Rightarrow \frac{x-7}{4} = \frac{y-4}{1} = \frac{z+1}{2}$$ **(1)**

[dividing denominator by –2]

48. Find the image of the point (1, 6, 3) on the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$. Also, write the equation of the line joining the given points and its image and find the length of segment joining given point and its image. **Delhi 2010C**

> Firstly, find the coordinates of foot of perpendicular Q. Then, find the image which is point T by using the fact that Q is the mid-point of line PT.

Let T be the image of the point P (1, 6, 3). Q is the foot of perpendicular PQ on the line AB.

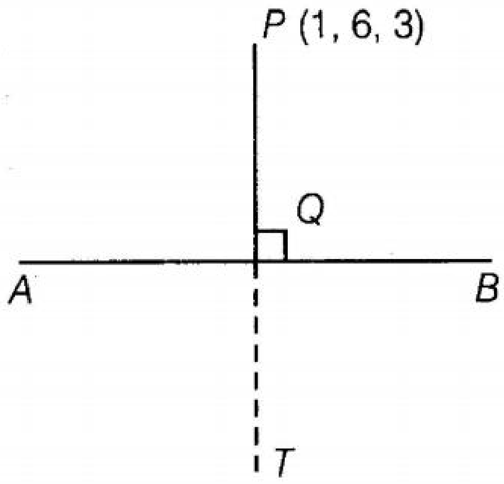
Given equation of line AB is

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} \quad \text{...(i)}$$

Let $$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = \lambda \quad \text{[say]}$$

$\Rightarrow$ $x = \lambda, y - 1 = 2\lambda, z - 2 = 3\lambda$

$\Rightarrow$ $x = \lambda, y = 2\lambda + 1, z = 3\lambda + 2$



Then, coordinates of

$$Q = (\lambda, 2\lambda + 1, 3\lambda + 2) \quad \text{...(ii)} \ \mathbf{(1)}$$

Now, DR's of line

$$PQ = (\lambda - 1, 2\lambda + 1 - 6, 3\lambda + 2 - 3)$$

$\Rightarrow$ DR's of $PQ = (\lambda - 1, 2\lambda - 5, 3\lambda - 1)$

Since, line $PQ \perp AB$.

Therefore, $a_1a_2 + b_1b_2 + c_1c_2 = 0$, where $a_1 = \lambda - 1, b_1 = 2\lambda - 5, c_1 = 3\lambda - 1$

and $a_2 = 1, b_2 = 2, c_2 = 3$

$\therefore\ 1(\lambda - 1) + 2(2\lambda - 5) + 3(3\lambda - 1) = 0$

$\Rightarrow \lambda - 1 + 4\lambda - 10 + 9\lambda - 3 = 0$

$\Rightarrow 14\lambda - 14 = 0$

$\Rightarrow \lambda = 1$ **(1)**

On putting $\lambda = 1$ in Eq. (ii), we get

$$Q(1, 2+1, 3+2) = (1, 3, 5)$$

Now, Q is the mid-point of PT.

Let coordinates of $T = (x, y, z)$

By using mid-point formula, **(1)**

$$Q = \text{mid-point of } P(1, 6, 3) \text{ and } T(x, y, z)$$

$$= \left(\frac{x+1}{2}, \frac{y+6}{2}, \frac{z+3}{2}\right)$$

$$\left[\because \text{mid-point} = \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right]$$

But $Q = (1, 3, 5)$

$$\therefore \left(\frac{x+1}{2}, \frac{y+6}{2}, \frac{z+3}{2}\right) = (1, 3, 5)$$

On equating corresponding coordinates, we get

$$\frac{x+1}{2}=1, \frac{y+6}{2}=3, \frac{z+3}{2}=5$$

$\Rightarrow$ $x=2-1, y=6-6, z=10-3$

$\Rightarrow$ $x=1, y=0, z=7$

$\therefore$ Coordinates of $T=(x, y, z)=(1, 0, 7)$

Hence, coordinates of image of point $P(1, 6, 3)$ is $T\ (1, 0, 7)$. **(1)**

Now, equation of line joining points $P(1, 6, 3)$ and $T(1, 0, 7)$ is

$$\frac{x-1}{1-1}=\frac{y-6}{0-6}=\frac{z-3}{7-3}$$

$$\Rightarrow \quad \frac{x-1}{0}=\frac{y-6}{-6}=\frac{z-3}{4} \quad \textbf{(1)}$$

Also, length of segment PT

$$=\sqrt{(1-1)^2+(6-0)^2+(3-7)^2}$$

$$=\sqrt{0+36+16}=\sqrt{52} \text{ units} \quad \textbf{(1)}$$

49. Write the vector equations of following lines and hence find the distance between them

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}, \quad \frac{x-3}{4}=\frac{y-3}{6}=\frac{z+5}{12}.$$

Delhi 2010

Given equations of lines are

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$$

and $$\frac{x-3}{4}=\frac{y-3}{6}=\frac{z+5}{12} \quad \textbf{(1)}$$

Now, the vector equation of lines are

$$\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda(2\hat{i}+3\hat{j}+6\hat{k}) \quad \ldots(i)$$

[$\because$ vector form of equation of line is

$$\vec{r}=\vec{a}+\lambda\vec{b}]$$

and $\vec{r}=(3\hat{i}+3\hat{j}-5\hat{k})+\mu(4\hat{i}+6\hat{j}+12\hat{k}) \ldots(ii)$

Here, $\vec{a_1}=\hat{i}+2\hat{j}-4\hat{k}$, $\vec{b_1}=2\hat{i}+3\hat{j}+6\hat{k}$

and $\vec{a_2}=3\hat{i}+3\hat{j}-5\hat{k}$, $\vec{b_2}=4\hat{i}+6\hat{j}+12\hat{k}$

Then, $\vec{a_2}-\vec{a_1}=(3\hat{i}+3\hat{j}-5\hat{k})-(\hat{i}+2\hat{j}-4\hat{k})$

$$=2\hat{i}+\hat{j}-\hat{k} \quad \ldots(iii)\ \textbf{(1)}$$

$$\vec{b_1}\times\vec{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 6\\ 4 & 6 & 12\end{vmatrix}$$

$$=\hat{i}(36-36)-\hat{j}(24-24)+\hat{k}(12-12)$$

$$=0\hat{i}-0\hat{j}+0\hat{k}=\vec{0} \quad \textbf{(1)}$$

$$\Rightarrow \quad \vec{b_1}\times\vec{b_2}=\vec{0}$$

$\Rightarrow$ Vector $\vec{b_1}$ is parallel to $\vec{b_2}$

[$\because$ if $\vec{a} \times \vec{b} = \vec{0}$, then $\vec{a} \parallel \vec{b}$]

As, two lines are parallel.

$\therefore \quad \vec{b} = (2\hat{i} + 3\hat{j} + 6\hat{k}) \quad$ ...(iv)

[since, DR's of given lines are proportional]**(1)**

Since, the two lines are parallel, we use the formula for shortest distance between two parallel lines.

We know that,

shortest distance, $d = \left| \dfrac{\vec{b} \times (\vec{a_2} - \vec{a_1})}{|\vec{b}|} \right| \quad$ ...(v)

From Eqs. (iii), (iv) and (v), we get

$$d = \left| \frac{(2\hat{i} + 3\hat{j} + 6\hat{k}) \times (2\hat{i} + \hat{j} - \hat{k})}{\sqrt{(2)^2 + (3)^2 + (6)^2}} \right| \quad \text{...(vi)}$$

Now, $(2\hat{i} + 3\hat{j} + 6\hat{k}) \times (2\hat{i} + \hat{j} - \hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}(-3 - 6) - \hat{j}(-2 - 12) + \hat{k}(2 - 6)$$

$$= -9\hat{i} + 14\hat{j} - 4\hat{k} \quad \textbf{(1)}$$

From Eq. (vi), we get

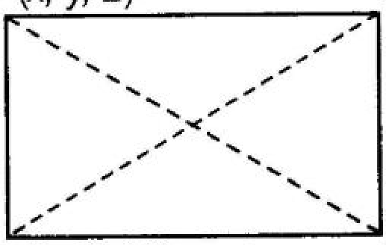
$$d = \left| \frac{-9\hat{i} + 14\hat{j} - 4\hat{k}}{\sqrt{49}} \right| = \frac{\sqrt{(-9)^2 + (14)^2 + (-4)^2}}{7}$$

$$\Rightarrow \quad d = \frac{\sqrt{81 + 196 + 16}}{7} = \frac{\sqrt{293}}{7} \text{ units} \quad \textbf{(1)}$$

50. The points $A(4, 5, 10)$, $B(2, 3, 4)$ and $C(1, 2, -1)$ are three vertices of parallelogram $ABCD$. Find the vector equations of sides AB and BC and also find coordinates of point D. HOTS; Delhi 2010

> The vector equation of a side of a parallelogram, when two points are given, is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$. Also, the diagonals of a rectangle intersect each other at mid-point.

Given points are $A(4, 5, 10)$, $B(2, 3, 4)$ and $C(1, 2, -1)$



We know that, two points vector form of line is given by

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) \quad \text{...(i)} \ \mathbf{(1)}$$

where, $\vec{a}$ and $\vec{b}$ are the position vector of points through which the line is passing through. Here, for line AB position vectors

are $\quad \vec{a} = \vec{OA} = 4\hat{i} + 5\hat{j} + 10\hat{k}$

and $\quad \vec{b} = \vec{OB} = 2\hat{i} + 3\hat{j} + 4\hat{k} \quad$ **(1)**

Using Eq. (i), the required equation of line AB is

$$\vec{r} = (4\hat{i} + 5\hat{j} + 10\hat{k}) + \lambda[2\hat{i} + 3\hat{j} + 4\hat{k} - (4\hat{i} + 5\hat{j} + 10\hat{k})]$$

$$\Rightarrow \vec{r} = (4\hat{i} + 5\hat{j} + 10\hat{k}) - \lambda(2\hat{i} + 2\hat{j} + 6\hat{k}) \quad \mathbf{(1)}$$

Similarly, vector equation of line BC, where $B(2, 3, 4)$ and $C(1, 2, -1)$ is

$$\vec{r} = (2\hat{i} + 3\hat{j} + 4\hat{k}) + \mu[\hat{i} + 2\hat{j} - \hat{k}$$

$- (2\hat{i} + 3\hat{j} + 4\hat{k})]$

$\Rightarrow \quad \vec{r} = (2\hat{i} + 3\hat{j} + 4\hat{k}) - \mu(\hat{i} + \hat{j} + 5\hat{k})$ **(1)**

We know that, mid-point of diagonal *BD*
= Mid-point of diagonal *AC*

[$\because$ diagonal of a parallelogram bisect each other]

$$\therefore \left(\frac{x+2}{2}, \frac{y+3}{2}, \frac{z+4}{2}\right) = \left(\frac{4+1}{2}, \frac{5+2}{2}, \frac{10-1}{2}\right)$$

(1)

On comparing corresponding coordinates, we get

$$\frac{x+2}{2} = \frac{5}{2}, \quad \frac{y+3}{2} = \frac{7}{2}$$

and $\dfrac{z+4}{2} = \dfrac{9}{2} \Rightarrow x = 3, y = 4$ and $z = 5$

Hence, coordinates of point
$D(x, y, z) = (3, 4, 5)$ **(1)**

and vector equations of sides *AB* and *BC* are

$\vec{r} = (4\hat{i} + 5\hat{j} + 10\hat{k}) - \lambda(2\hat{i} + 2\hat{j} + 6\hat{k})$ and

$\vec{r} = (2\hat{i} + 3\hat{j} + 4\hat{k}) - \mu(\hat{i} + \hat{j} + 5\hat{k})$,

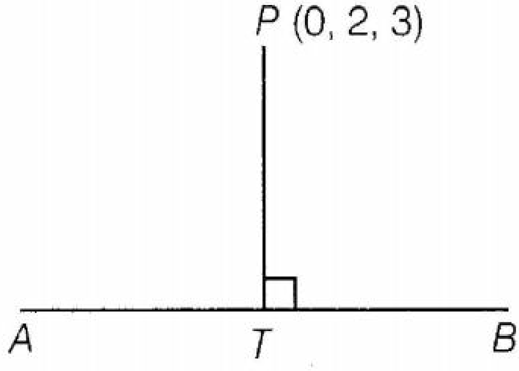
respectively.

51. Find the coordinates of foot of perpendicular drawn from the point (0, 2, 3) on the line $\dfrac{x+3}{5} = \dfrac{y-1}{2} = \dfrac{z+4}{3}$. Also, find the length of perpendicular. Delhi 2009C

Given equation of line is

$$\frac{x+3}{5}=\frac{y-1}{2}=\frac{z+4}{3}$$

and given point is $P(0, 2, 3)$, let foot of perpendicular PT is T.



Now, $$\frac{x+3}{5}=\frac{y-1}{2}=\frac{z+4}{3}=\lambda \quad \text{[say]}$$ **(1)**

$\Rightarrow$ $x=5\lambda-3, y=2\lambda+1, z=3\lambda-4$

$\therefore$ Coordinates of point T are

$(5\lambda-3, 2\lambda+1, 3\lambda-4)$ **(1)**

DR's of line

$$PT=(5\lambda-3-0, 2\lambda+1-2, 3\lambda-4-3)$$
$$=(5\lambda-3, 2\lambda-1, 3\lambda-7)$$ **(1)**

Since, $PT \perp AB$

Therefore, $a_1a_2+b_1b_2+c_1c_2=0$

where, $a_1 = 5\lambda - 3,\ b_1 = 2\lambda - 1,\ c_1 = 3\lambda - 7$

and $\quad a_2 = 5, b_2 = 2, c_2 = 3$ **(1)**

$\therefore 5(5\lambda - 3) + 2(2\lambda - 1) + 3(3\lambda - 7) = 0$

$\Rightarrow \quad 25\lambda - 15 + 4\lambda - 2 + 9\lambda - 21 = 0$

$\Rightarrow \quad 38\lambda - 38 = 0 \Rightarrow 38\lambda = 38$

$\Rightarrow \quad \lambda = 1$ **(1)**

$\therefore$ The foot of perpendicular

$$T = (5\lambda - 3, 2\lambda + 1, 3\lambda - 4)$$

$$= (2, 3, -1) \qquad [\text{put } \lambda = 1]$$ **(1/2)**

Also, length of perpendicular, PT = Distance between points P and T

$\Rightarrow PT = \sqrt{(0-2)^2 + (2-3)^2 + (3+1)^2}$

$\left[\because \text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\right]$

$= \sqrt{4 + 1 + 16} = \sqrt{21}$ units **(1/2)**

52. Find the perpendicular distance of the point (2, 3, 4) from the line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$.
Also, find coordinates of foot of perpendicular. Delhi 2009C

Do same as Que. 51.

$\left[\textbf{Ans. } \text{Perpendicular distance} = \text{Distance} \quad \text{coordinates of foot} = \left(\frac{170}{49}, \frac{78}{49}, \frac{60}{49}\right)\right]$